

# TRINOMIALS DEFINING QUINTIC NUMBER FIELDS

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**ABSTRACT.** Given a quintic number field  $K/\mathbb{Q}$ , we study the set of irreducible trinomials, polynomials of the form  $x^5 + ax + b$ , that have a root in  $K$ . We show that there is a genus four curve  $C_K$  whose rational points are in bijection with such trinomials. This curve  $C_K$  maps to an elliptic curve defined over a number field, and using this map, we are able (in some cases) to determine all the rational points on  $C_K$  using elliptic curve Chabauty.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Number fields defined by trinomials, polynomials of the shape,  $f(x) = ax^n + bx + c$ , have a long history. Two such trinomials  $f_1(x)$  and  $f_2(x)$  are equivalent if  $f_2(x) = \alpha f_1(\beta x)$  for some  $\alpha, \beta \in \mathbb{Q}^\times$ . If  $n = 5$ , the Galois group of a trinomial is contained in  $F_{20}$ , the Frobenius group of order 20, if and only if it is equivalent to one of the form

$$(4u^2 + 16)x^5 + (5u^2 - 5)x + (4u^2 + 10u + 6),$$

(see [22] section 189, [14], pages 90-91, and [19]). Moreover, the Galois group is contained in the dihedral group of order 10 if and only if  $u = t - 1/t$ .

Malle [13] determined the family of degree 6 trinomials with Galois group contained in  $S_5$ , namely  $(125 - u)x^6 + 12u(u + 3)^2x + u(u - 5)(u + 3)^2$ . Many transitive subgroups of  $S_6$  occur as Galois groups of degree 6 trinomials, including for example  $\mathbb{Z}/6\mathbb{Z}$  which is the Galois group of  $f(x) = x^6 + 133x + 209$ . (This is the only sextic trinomial, up to equivalence, with this Galois group, as shown in [2].)

In 1969, Trinks discovered the trinomial  $f(x) = x^7 - 7x + 3$  which was proven (by Matzat) to have Galois group  $\mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{PSL}_2(\mathbb{Z}/7\mathbb{Z})$ , the unique finite simple group of order 168. In 1979, Erbach, Fischer, and McKay [9] found a second such trinomial,  $f(x) = x^7 - 154x + 99$ . Finally, in 1999, Elkies and Bruin [5] proved that the set of such trinomials with Galois group  $\mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})$  is in bijection with the rational points on a genus 2 curve, and there are exactly four equivalence classes of such trinomials (including, of course, the two mentioned above). They also studied degree 8 trinomials with Galois group contained in  $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})$  and found four examples, including  $x^8 + 324x + 567$ , whose Galois group is  $\mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})$ . The classification of finite simple groups puts a lot of restrictions on the Galois group that an irreducible trinomial  $x^n + ax^s + b$  can have. In particular (see [11]), if  $n > 11$  is prime then the Galois group  $G$  is either solvable, isomorphic to  $A_n$  or  $S_n$ , or  $n = 2^e + 1$  is a Fermat prime and  $\mathrm{GL}_2(\mathbb{F}_{2^e}) \subseteq G \subseteq \mathbb{F}_{2^e}^\times \rtimes \mathrm{GL}_2(\mathbb{F}_{2^e})$ .

In this paper, we instead study the following question. Fix a degree 5 number field  $K/\mathbb{Q}$ . What are all the irreducible trinomials  $x^5 + ax + b$  that have a root

in  $K$ ? We make progress on answering this question. The following result is a summary of what we accomplish.

**Theorem 1.** *Let  $K = \mathbb{Q}[\alpha]$  be a degree 5 extension of  $\mathbb{Q}$ , and let  $g(x) \in \mathbb{Q}[x]$  be the minimal polynomial of  $\alpha$ . Then,*

- *There is a genus four curve  $C_K/\mathbb{Q}$  whose rational points  $C_K(\mathbb{Q})$  are in bijection with equivalence classes of irreducible trinomials  $f(x) = x^5 + ax + b$  so that  $K \cong \mathbb{Q}[x]/(f(x))$ .*
- *If  $L$  is the smallest field over which  $g(x)$  factors into a quadratic and a cubic in  $L[x]$ , there is an elliptic curve  $E/L$  and a degree 2 map  $\phi : C_K \rightarrow E$  defined over  $L$ . The curve  $E$  has  $j$ -invariant  $-25/2$ .*
- *Assuming that the Galois closure of  $K$  over  $\mathbb{Q}$  has Galois group  $F_{20}$ ,  $A_5$  or  $S_5$  and  $K/\mathbb{Q}$  is unramified at 2 and 5, the root number of  $E/L$  is even.*
- *There is a degree 60 map  $\psi : E \rightarrow \mathbb{P}^1$  with the property that if  $P \in C_K(\mathbb{Q})$ , then  $\psi \circ \phi(P) \in \mathbb{P}^1(\mathbb{Q})$ .*

**Remark.** *Since  $C_K$  has genus four, Faltings's theorem [10] proves that  $C_K(\mathbb{Q})$  is finite, and so there are only finitely many irreducible trinomials (up to equivalence) with a root in  $K$ . Moreover, Caporaso, Harris and Mazur showed [6] that the weak Lang conjecture (that the set of rational points on a variety of general type is not Zariski dense) implies a bound on the number of rational points on a curve of genus  $g$  depending only on  $g$ .*

The map  $\psi : E \rightarrow \mathbb{P}^1$  above provides a convenient way to study the rational points on  $C_K$ . In particular, if it is possible to find generators for  $E(L)$ , then the method of elliptic curve Chabauty (see [4]) allows one to provably determine all of the rational points on  $C_K$ .

**Theorem 2.** (1) *Let  $f(x) = x^5 - 5x + 12$ , and  $\alpha$  be a root of  $f(x)$ . The field  $K = \mathbb{Q}[\alpha]$  is a number field whose Galois closure has dihedral Galois group. Up to equivalence, the only trinomial with a root in  $K$  is  $f(x)$ .*  
 (2) *Assume the generalized Riemann hypothesis. Let  $K = \mathbb{Q}[\sqrt[5]{18}]$ . Up to equivalence, the only trinomials with a root in  $K$  are the following:*

$$\begin{aligned} f_1(x) &= x^5 - 18 \\ f_2(x) &= x^5 - 324 \\ f_3(x) &= x^5 - 24 \\ f_4(x) &= x^5 - 432 \\ f_5(x) &= x^5 + 750x + 3750. \end{aligned}$$

*The fifth trinomial above is mentioned in Example 3.1 of [20], page 385.*

**Remark.** *The main difficulty in applying elliptic curve Chabauty is finding generators for  $E(L)$ , because point searching over number fields of degree 10 is very difficult. The easiest way to arrange this is to find lots of points on  $E$  by taking images of rational point on  $C_K$ . A priori, elliptic curve Chabauty can be successful if  $\text{rank } E(L) < [L : \mathbb{Q}]$ . However, the map  $C_K/\mathbb{Q} \rightarrow E/L$  gives a map from  $\text{Jac}(C) \rightarrow \text{Res}_{L/\mathbb{Q}}(E)$ . If the rank of  $\text{Jac}(C)(\mathbb{Q})$  is greater than 3, then elliptic curve Chabauty will not succeed. (This was explained to the authors by Nils Bruin.) In the case that  $K = \mathbb{Q}[\alpha]$ , where  $\alpha$  is a root of  $f(x) = x^5 + 2x^4 - 11x^3 - 17x^2 + 26x + 55$ , there are five obvious rational points on  $C_K$  and (assuming GRH) the rank of  $\text{Jac}(C)$*

is equal to that of  $\text{Res}_{L/\mathbb{Q}}(E)$  which is equal to 4 (as confirmed by a long 2-descent computation). In this case, elliptic curve Chabauty will not be successful.

**Remark.** The largest size of  $C_K(\mathbb{Q})$  we have found is 8. In particular, if  $K = \mathbb{Q}[\alpha]$ , where  $\alpha$  is a root of  $f_1(x) = x^5 + 75x + 105$ , then  $K$  also contains roots of

$$\begin{aligned} f_2(x) &= x^5 - 75x + 465 \\ f_3(x) &= x^5 - 1125x + 3825 \\ f_4(x) &= x^5 - 2025x + 65205 \\ f_5(x) &= x^5 + 2025x + 10665 \\ f_6(x) &= x^5 - 10125x + 83025 \\ f_7(x) &= x^5 + 28125x - 39375 \\ f_8(x) &= x^5 - 3410625x + 86685375. \end{aligned}$$

The example above raises the following question: what is the largest number of rational points  $C_K$  can have? Are there families of quintic fields that each have many trinomials?

In [20], Spearman and Williams gave the example that if  $r \neq 0, \pm 1$  is rational, and  $\alpha^5 = r^3(r+1)(r-1)^4$ , then  $\mathbb{Q}[\alpha]$  also contains the roots of  $x^5 + ax + b$ , where

$$\begin{aligned} a &= \frac{-80r(r^2 - 1)(r^2 + r - 1)(r^2 - 4r - 1)}{(r^2 + 1)^4} \\ b &= \frac{-32r(r^2 - 1)(r^4 + 22r^3 - 6r^2 - 22r + 1)}{(r^2 + 1)^4}. \end{aligned}$$

We give a different one-parameter family of number fields that each have (at least) two trinomials  $x^5 + ax + b$  in them with  $a$  and  $b$  both nonzero.

**Theorem 3.** Let  $a \in \mathbb{Q}$  with  $a \neq -8$ . Suppose that

$$f(x) = (4a + 32)x^5 + (-5a^2 + 5a)x - a^3 + a^2$$

is irreducible. Let  $\alpha$  be a root of  $f(x)$ . Then the number

$$\frac{1}{a^2 + 4a - 8} \left[ \frac{4a^2 + 16a - 128}{a^2 - a} \alpha^4 + \frac{8a + 64}{a} \alpha^3 + (-2a - 16) \alpha^2 + (2a + 4) \alpha - 4a + 16 \right]$$

is a root of

$$(a^3 + 7a^2 - 8a)x^5 + (10a^2 + 115a - 125)x + 2a^2 - 76a - 250.$$

(We omit the proof of this theorem, as it is a straightforward calculation.)

In Section 2, we recall background on trinomials, the canonical ring of a curve, root numbers of elliptic curves, and elliptic curve Chabauty. In Section 3, we prove parts 1 and 2 of Theorem 1. In Section 4, we prove parts 3 and 4 of Theorem 1. In Section 5, we give prove Theorem 2. In Section 6, we discuss a K3 surface  $X$  whose rational points correspond to number fields  $K$  that have more than one trinomial (up to equivalence) defining them. Theorem 3 results from a rational curve on this surface.

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## 2. BACKGROUND

If  $f_1(x) = x^5 + ax + b$  and  $f_2(x) = x^5 + cx + d$  are two trinomials, we say that  $f_1$  and  $f_2$  are equivalent if there is a rational number  $\alpha \neq 0$  so that  $\frac{1}{\alpha^5}f_1(\alpha x) = f_2(x)$ . If  $b$  and  $d$  are not zero, we have that  $f_1(x)$  is equivalent to  $f_2(x)$  if and only if  $a^5/b^4 = c^5/d^4$ . This motivates the definition of the parameter  $t = a^5/b^4$ , defined on trinomials  $f_1(x) = x^5 + ax + b$ . If  $a \neq 0$  and  $b \neq 0$ , note that  $f_1(x)$  is equivalent to  $x^5 + tx + t$ .

If  $C$  is a curve of genus  $g \geq 2$  with  $\Omega$  the sheaf of holomorphic differential 1-forms on  $C$ , the canonical ring is

$$R(C) = \bigoplus_{d=0}^{\infty} H^0(C, \Omega^{\otimes d}).$$

We have that  $C \cong \text{Proj } R(C)$ , as the canonical divisor is ample. We will use the canonical ring of our genus four curves  $C_K$  and an automorphism  $\tau : C \rightarrow C$  of order 2 to construct the curve quotient  $C_K/\langle \tau \rangle$ .

If  $E/K$  is an elliptic curve defined over a number field  $K$ , the root number of  $E/K$ ,  $w_{E/K}$  is defined in terms of local  $\epsilon$ -factors corresponding to representations of local Weil-Deligne groups. (See [7] for more details.) The number  $w_{E/K}$  is conjecturally the sign of the functional equation of the  $L$ -function  $L(E/K, s)$ . Since the Birch and Swinnerton-Dyer conjecture predicts that the rank of  $E(K)$  is equal to the order of vanishing of  $L(E/K, s)$  at  $s = 1$ , we can conjecturally predict the parity of the rank of  $E/K$  by determining  $w_{E/K}$ . If  $E/K$  is an elliptic curve and  $\rho : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{GL}_n(\mathbb{C})$  is an Artin representation, we define  $L(E \otimes \rho, s)$  to be the  $L$ -function of  $\text{tw}_{\rho}(E)$ , the twist of  $E$  by  $\rho$  (see [7], Section 3 for details about this definition).

The technique we will use for provably finding all the rational points on our genus four curves is elliptic curve Chabauty. This technique was developed theoretically by Nils Bruin (see [4]) and implemented in Magma. The setup for elliptic curve Chabauty is the following. Given a curve  $C/\mathbb{Q}$ , it is necessary to have a map  $\Phi : C \rightarrow \mathbb{P}^1$  defined over  $\mathbb{Q}$  that factors as  $\Phi = \pi \circ \varphi$ , where  $\pi : C \rightarrow E$  is a map to an elliptic curve defined over a number  $K$  field, and  $\varphi : E \rightarrow \mathbb{P}^1$ . Elliptic curve Chabauty is sometimes able to show that  $\{P \in E(K) : \varphi(P) \in \mathbb{P}^1(\mathbb{Q})\}$  is finite and determine its elements.

## 3. THE CURVE $C_K$

Fix a degree five number field  $K/\mathbb{Q}$  with  $K = \mathbb{Q}[\alpha]$ . Trinomials with a root in  $K$  are in bijection with elements  $\beta = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$  with the property that the characteristic polynomial of the linear transformation  $T_{\beta} : K \rightarrow K$  given by  $T_{\beta}(x) = \beta x$  has the form  $x^5 + rx + s$  for  $r, s \in \mathbb{Q}$ . By representing the transformation  $T_{\beta}$  in terms of the basis  $\{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$ , we obtain three equations involving  $a, b, c, d$ , and  $e$  that express this condition. For the rest of this section, we will focus on the important special case that  $\alpha$  is a root of  $x^5 + tx + t$ .

**Proposition 4.** *Suppose that  $K = \mathbb{Q}[\alpha]$ , where  $\alpha$  is a root of  $x^5 + tx + t$ . Then the trinomials in  $\mathbb{Q}[x]$  with a root in  $K$  (up to equivalence) are in bijection with the*

rational points on  $C_K(\mathbb{Q})$ , where  $C_K \subseteq \mathbb{P}^3$  is defined by

$$\begin{aligned} & -5a^2 + 50ab + 32tbd + 16tc^2 + 40tcd = 0 \\ & -10a^3 + 25a^2b - 125a^2c - 160tacd - 100tad^2 \\ & + 64tb^2c + 80tb^2d + 80tbc^2 - 64t^2cd^2 - 48t^2d^3 = 0. \end{aligned}$$

*Proof.* Applying the procedure above gives the equation  $4te = 5a$  from the assumption that the coefficient of  $x^4$  in the characteristic polynomial of  $T_\beta$  is zero. Plugging this into the equations that result from the vanishing of the coefficients of  $x^3$  and  $x^2$  yield the equations above. Finally, if  $(a_1 : b_1 : c_1 : d_1) = \lambda(a_2 : b_2 : c_2 : d_2)$  are two equivalent points on  $C_K$ , the corresponding trinomials are related by  $f_1(x) = \lambda^5 f_2(x/\lambda)$ , and are hence equivalent.  $\square$

**Remark.** The curve  $C_K$  is isomorphic over the Galois closure of  $K/\mathbb{Q}$  to the curve  $B$  defined by

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 0 \\ x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 &= 0, \end{aligned}$$

via the map  $x_i = a + b\alpha_i + c\alpha_i^2 + d\alpha_i^3 + e\alpha_i^4$ , where  $\alpha_1, \dots, \alpha_5$  are the conjugates of  $\alpha$ . This curve is known as Bring's curve after the work of Erland Samuel Bring (in 1786, as reported by Felix Klein on page 157 of [12]) on reducing a general quintic to one of the form  $x^5 + ax + b$ . This curve also arose in the work of A. Wiman (see [8]).

The curve  $B$  has 120 automorphisms, and this is the largest number possible for a curve of genus 4. Moreover, any genus 4 curve with 120 automorphisms is isomorphic (over  $\mathbb{C}$ ) to  $B$  (see [3]). The Jacobian of  $B$  is isogenous to  $E_0^4$ , where  $E_0 : y^2 = x^3 - 675x - 79650$  (see Section 8.3.2 of [18]).

To obtain equations for the map from  $C_K$  to an elliptic curve  $E$ , we first compute an automorphism of  $\tau$  of  $C_K$ . This automorphism is obtained as a composition of  $C_K \rightarrow B \rightarrow B \rightarrow C_K$ , where the map from  $B \rightarrow B$  is  $(x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_2 : x_1 : x_3 : x_4 : x_5)$ . This automorphism of  $C_K$  is the restriction to  $C_K$  of an automorphism of  $\mathbb{P}^3$  and hence can be represented by a  $4 \times 4$  matrix. This automorphism is defined over the smallest field  $L = \mathbb{Q}[z]$  over which  $f(x)$  factors as  $g(x)h(x)$ , where  $g(x)$  has the roots  $\alpha_1$  and  $\alpha_2$ . In general,  $L/\mathbb{Q}$  has degree 10, and  $z$  is a root of the polynomial

$$x^{10} - 3tx^6 - 11tx^5 - 4t^2x^2 + 4t^2x - t^2.$$

We have obtained the entries of the matrix giving the automorphism as elements of  $L[t]$  - they are too cumbersome to reproduce here.

Next, we will give a method to compute the quotient curve  $C_K/\langle\tau\rangle$ . This procedure will work on any non-hyperelliptic genus 4 curve with an involution  $\tau$  so that the quotient  $C_K/\langle\tau\rangle$  has genus 1. Identify  $C_K$  with its canonical embedding in  $\mathbb{P}^3$ . By Proposition 2.6 of [15] (page 207), the canonical image of a non-hyperelliptic genus 4 curve is the complete intersection of a quadric and a cubic,  $C_K : p_2(a, b, c, d) = p_3(a, b, c, d) = 0$ . We consider the action of  $\tau$  on  $H^0(C, \Omega)$ , the 4-dimensional space of holomorphic 1-forms. It is well-known (see

Theorem 7.5 on page 27 of [3]) that the subspace of  $H^0(C, \Omega)$  fixed by  $\tau$  is equal to the genus of the quotient curve. Under the assumption that  $C_K/\langle\tau\rangle$  has genus 1, it follows that  $\tau$  (thought of as a linear map  $H^0(C, \Omega) \rightarrow H^0(C, \Omega)$ ) has a one-dimensional 1-eigenspace, and hence a 3-dimensional  $-1$ -eigenspace. In general, we let  $H^0(C, \Omega^{\otimes i})^+$  (respectively  $H^0(C, \Omega^{\otimes i})^-$ ) denote the  $+$  and  $-$  eigenspaces of  $\tau$  acting on  $H^0(C, \Omega^{\otimes i})$ . Let  $\{e_1\}$  be a basis for  $H^0(C, \Omega)^+$  and  $\{e_2, e_3, e_4\}$  be a basis for  $H^0(C, \Omega)^-$ .

**Proposition 5.** *Assume the notation above. The morphism  $\phi : C \rightarrow \mathbb{P}^2$  given by  $\phi(a : b : c : d) = (e_2 : e_3 : e_4)$  is the quotient map. The image of  $\phi$  is a cubic curve.*

**Remark.** *The authors have computed the coefficients of the plane cubic as elements of  $\mathbb{Q}[z, t]$ .*

*Proof.* If the coefficient of  $e_1^2$  in  $p_2$  is zero, then  $e_1$  can be solved for in terms of  $e_2, e_3$  and  $e_4$ , which makes the map  $\phi : C \rightarrow \mathbb{P}^2$  have degree 1. This is impossible, however, because  $\phi(P) = \phi(\tau(P))$ . It follows that the coefficient of  $e_1^2$  in  $p_2$  is nonzero and this implies that  $\phi$  has degree 2.

We have that  $\dim H^0(C, \Omega^{\otimes 2})^+ = 7$  and is spanned by  $e_1^2, e_2^2, e_3^2, e_4^2, e_2e_3, e_2e_4$  and  $e_3e_4$ . On the other hand  $\dim H^0(C, \Omega^{\otimes 2})^- = 3$  and is spanned by  $e_1e_2, e_1e_3$  and  $e_1e_4$ . Since the automorphism  $\tau$  must map  $p_2$  to a  $\lambda p_2$  for some  $\lambda$ , and since  $p_2$  must be irreducible (since  $C$  is contained in a unique quadric surface), it follows that  $\tau(p_2) = p_2$  (because if  $\tau(p_2) = -p_2$ , then  $e_1$  would be a factor of  $p_2$ ). It follows that  $p_2 = c_1e_1^2 + c_2e_2^2 + c_3e_3^2 + c_4e_4^2 + c_5e_2e_3 + c_6e_2e_4 + c_7e_3e_4$ . This implies that if  $e_2, e_3$  and  $e_4$  are all zero, then  $e_1 = 0$ , and so  $\phi$  is a morphism. Since this morphism is invariant under the action of  $\tau$ , it factors  $C \rightarrow C/\langle\tau\rangle \rightarrow \text{im } \phi$ . The map from  $C/\langle\tau\rangle \rightarrow \text{im } \phi$  has degree 1, and must therefore be an isomorphism.

Decompose the polynomial  $p_3 = p_3^+ + p_3^-$ , where  $\tau(p_3^\pm) = \pm p_3^\pm$ . Note that  $p_3^-$  is nonzero, since if  $p_3 = p_3^+$ , then  $e_1|p_3^+$  and this implies that  $C$  is reducible. Consider the 14 polynomials  $p_3^-, e_2p_2, e_3p_2, e_4p_2$  and  $e_2^ie_3^je_4^k$  with  $i + j + k = 3$ . All of these lie in  $H^0(C, \Omega^{\otimes 3})^-$ , but  $\dim H^0(C, \Omega^{\otimes 3})^-$  is 13. It follows that these polynomials are linearly dependent. This implies that the image of  $\phi : C \rightarrow \mathbb{P}^2$  is a cubic curve, as desired.  $\square$

#### 4. THE ELLIPTIC CURVE $E/L$

The assumption that  $K$  is defined by  $x^5 + tx + t$  forces  $(0 : 1 : 0 : 0)$  to be a rational point on the curve  $C$ . The image of this rational point on the plane cubic turns out to be  $(0 : 1 : 0)$ .

By putting the plane cubic in Weierstrass form, we find that (for  $t \neq -3125/256$ ),

$$E : y^2 = x^3 - 675\beta^2x - 79650\beta^3,$$

where  $\beta = \frac{16}{5}z^9 + \frac{8}{5}z^8 + \frac{4}{5}z^7 - 3z^6 - \frac{48}{5}tz^5 - 40tz^4 - 20tz^3 + \frac{13}{5}tz^2 + (-64/5t^2 + 29t)z + \frac{32}{5}t^2$ . (If  $t = -3125/256$ , then  $x^5 + tx + t$  has a repeated linear factor.) This curve is a quadratic twist of  $E_0 : y^2 = x^3 - 675x - 79650$ , which has conductor 50,  $j(E_0) = -25/2$ , and is one of four elliptic curve over  $\mathbb{Q}$  (up to quadratic twist) with a rational 15-isogeny. The extension  $L[\sqrt{\beta}]/L$  is a degree 2 extension inside the splitting field of  $x^5 + tx + t$ . In particular, over the field  $L[\sqrt{\beta}]$ ,  $x^5 + tx + t$  has two linear factors.

**Proposition 6.** *If  $t \neq 0$ ,  $t \neq 3125/144$ ,  $E(L)$  has positive rank.*

*Proof.* If  $t = 3125/144$ , the point  $(0 : 1 : 0)$  is a flex on the plane cubic found in the previous section. Otherwise, there is a second point  $Q$  on the tangent line to the plane cubic at  $(0 : 1 : 0)$ . Call  $P$  the image of this point on  $E$ . A straightforward computation shows that if  $3P = (0 : 1 : 0)$  or  $5P = (0 : 1 : 0)$ , then  $t = 0$  or  $3125/144$ . We will show that in all other cases,  $P$  has infinite order.

Because  $E$  is isomorphic to  $E_0 : y^2 = x^3 - 675x - 79650$  over  $L[\sqrt{\beta}]$ , if  $P$  has order  $k$ , then  $E_0$  must have a point of order  $k$  defined over a degree 20 extension. Consider the mod  $\ell$  Galois representation attached to  $E_0$ . If this representation is surjective, then any field over which  $E_0$  acquires an  $\ell$ -torsion point must have degree a multiple of  $\ell^2 - 1$ . This is  $> 20$  for any  $\ell > 5$ . Andrew Sutherland has verified the surjectivity of the mod  $\ell$  Galois representation attached to  $E_0$  for  $5 < \ell < 80$  (see [21]). Moreover, it is known (by work of Serre, see [17]) that if  $\ell > 37$  and the mod  $\ell$  Galois representation is not surjective, there is a quadratic character  $\chi$  unramified at primes not dividing  $50 = N(E_0)$  for which  $a_p \equiv 0 \pmod{\ell}$  for any  $p \nmid N$  for which  $\chi(p) = -1$ . It is easy to see (by considering  $p = 3, 7$  and  $11$ ) that no such  $\chi$  exists. Thus, the mod  $\ell$  Galois representation is surjective if  $\ell > 5$ . It follows that  $P$  is not a torsion point unless  $t = 0$  or  $t = 3125/144$ .  $\square$

The above results raises the hope that for a generic value of  $t$ , the rank of  $E(L)$  might be one. This is (probably) false for many values of  $t$  as evidenced by the following result.

**Proposition 7.** *Suppose that  $K/\mathbb{Q}$  is unramified at 2 and 5 and the Galois of  $x^5 + tx + t$  is isomorphic to  $F_{20}$ ,  $A_5$  or  $S_5$ . Then the root number of  $w_{E/L} = 1$ .*

**Remark.** *If  $t = -45/4$ , then the Galois group of  $x^5 + tx + t$  is isomorphic to  $S_5$  and there are at least 5 trinomials with a root in  $K$ . The number field  $K/\mathbb{Q}$  is ramified at 2 and 5, and the root number  $w_{E/L} = -1$ , showing that the hypothesis that 2 and 5 are unramified in  $K/\mathbb{Q}$  is necessary.*

Assuming that the  $L$ -function of  $E/L$  has an analytic continuation and a functional equation with root number  $w_{E/L} = 1$ , the Birch and Swinnerton-Dyer conjecture predicts that the rank of  $E(L)$  is even.

*Proof.* We will use Corollary 2 of [7] which states that if  $\rho$  is a self-dual Artin representation and  $E$  is an elliptic curve defined over  $\mathbb{Q}$  with conductor coprime to that of  $\rho$ , then

$$w_{\text{tw}_\rho}(E) = w_E^{\dim \rho} \text{sign}(\alpha_\rho) \left( \frac{\alpha_\rho}{N} \right),$$

where  $\mathbb{Q}(\sqrt{\alpha_\rho})$  is the fixed field of  $\det \rho$ . Suppose  $E/\mathbb{Q}$  is an elliptic curve and  $K/\mathbb{Q}$  is a number field with  $K = \mathbb{Q}[\alpha]$ , where  $\alpha$  has the minimal polynomial  $f(x)$ . Then we have  $L(E/K, s) = L(E \otimes \rho, s)$ , where  $\rho : \text{Gal}(K/\mathbb{Q}) \rightarrow S_n \rightarrow \text{GL}_n(\mathbb{C})$  is the Artin representation obtained from the permutation character giving the action of  $\text{Gal}(K/\mathbb{Q})$  on the roots of  $f$ . We apply this with  $E_0 : y^2 = x^3 - 675x - 79650$ . We get that  $L(E, s) = L(E_0/L[\sqrt{\beta}], s)/L(E_0/L, s)$ .

Suppose the Galois group of  $x^5 + tx + t$  over  $\mathbb{Q}$  is  $S_5$ . In this case,  $L(E, s) = L(E_0 \otimes \rho_3, s)L(E_0 \otimes \rho_7, s)$ , where  $\rho_3$  is the irreducible 4-dimensional representation with  $\text{tr } \rho_3(\sigma) = -2$  when  $\sigma$  is a transposition. Here  $\rho_7$  is the irreducible 6-dimensional representation. Both  $\det \rho_3$  and  $\det \rho_7$  are non-trivial, but there is a unique quadratic subfield of the Galois closure, and so each of  $L(E_0 \otimes \rho_3, s)$  and

$L(E_0 \otimes \rho_7, s)$  has root number

$$\text{sign}(\alpha_p) \left( \frac{\alpha_p}{N} \right).$$

Thus, their product has root number 1, so this is the root number of  $L(E, s)$ .

In the  $A_5$  case, the representations  $\rho_3$  and  $\rho_7$  have trivial determinant (since there are no quadratic subfields) and are no longer irreducible. However, the same argument applies. In the case that the Galois group is  $F_{20}$ , the  $L$ -function factors as  $L(E_0 \otimes \rho_1, s)L(E_0 \otimes \bar{\rho}_1, s)L(E_0 \otimes \rho_2, s)^2$ . Here  $\rho_1$  and  $\bar{\rho}_1$  are linear and the product of  $L(E_0 \otimes \rho_1, s)L(E_0 \otimes \bar{\rho}_1, s)$  has root number 1. The representation  $\rho_2$  is self-dual and so  $L(E_0 \otimes \rho_2, s)^2$  has root number 1.  $\square$

The final part of Theorem 1 remains to be established. We have the map  $\phi : C_K \rightarrow E$ . The easiest way to find a map  $\rho : C_K \rightarrow \mathbb{P}^1$  which can be written as  $\rho = \psi \circ \phi$  is to have  $\rho$  be the quotient map by the entire automorphism group of  $C_K$ . As mentioned above  $C_K$  is isomorphic to  $B$ , and it easy to see that the quotient map  $B \rightarrow B/\text{Aut } B$  is given by  $\xi : B \rightarrow \mathbb{P}^1$ , where  $\xi(x_1 : x_2 : x_3 : x_4 : x_5) = (\alpha^5 : \beta^4)$ , where

$$(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) = x^5 + \alpha x + \beta.$$

Composing  $\xi$  with the isomorphisms mapping  $B$  to and from  $C_K$  gives that  $\rho : C_K \rightarrow \mathbb{P}^1$  is given by  $\phi(a : b : c : d) = (\gamma^5 : \delta^4)$ , where  $x^5 + \gamma x + \delta$  is the minimal polynomial of  $\beta = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$  (in the notation of Section 3). This map  $\rho$  has degree 120, and is the quotient map  $C_K \rightarrow C_K/\text{Aut } C_K$ . Therefore, we can factor it as  $C_K \rightarrow C_K/\langle \tau \rangle \rightarrow C_K/\text{Aut } C_K \cong \mathbb{P}^1$ , and the map  $\psi : C_K/\langle \tau \rangle \rightarrow \mathbb{P}^1$  is the degree 60 map we seek. This concludes the proof of Theorem 1.

## 5. EXAMPLES

In this section, we prove Theorem 2.

*Proof.* The polynomial  $x^5 - 5x + 12$  is equivalent to  $x^5 + tx + t$ , where  $t = -3125/20736$ . The Galois group of  $x^5 - 5x + 12$  is  $D_{10}$ , and so  $L$ , the smallest field over which  $f(x)$  factors as the product of a quadratic and a cubic is  $K = \mathbb{Q}[z]$ , where  $z^5 - 5z + 12 = 0$ . The elliptic curve  $E$  is isomorphic to  $y^2 = x^3 - 675x - 79650$  over  $K[\sqrt{-10}]$ , the splitting field of the quintic. It follows that  $E$  is the  $-10$  quadratic twist of  $E_0$ , and our model for  $E$  can be given by  $E : y^2 = x^3 - x^2 - 833x + 109537$ . Magma's routines for doing 2-descents on curves over number fields determines that the rank of  $E$  is at most 2, and finds two generators (one with  $x = 10z^3 + 10z^2 - 30z + 47$  and the second with  $x = -10z^3 - 10z^2 + 30z - 13$ ).

The most difficult part of the computation is determining equations for the map  $\psi : E \rightarrow \mathbb{P}^1$ . We have in hand equations for the map  $\rho : C_K \rightarrow \mathbb{P}^1$  and equations for the map from  $C_K \rightarrow E$  and using this we can create a number of pairs  $(P, Q)$  where  $P \in E(K)$  and  $Q = \psi(P) \in \mathbb{P}^1(K)$  by taking a point  $P \in E(K)$ , computing the two preimages of  $P$  on  $C_K$ , and computing the image of one of these points under  $\rho$ .

Based on a few examples, we observe that  $\psi(-P) = \psi(P)$  and therefore, we expect that  $\psi(x, y)$  has the form  $\frac{A(x)}{B(x)}$ , where  $\deg A, B \leq 30$ . We use linear algebra to compute  $A(x)$  and  $B(x)$  and we verify that  $\psi(x, y) = \frac{A(x)}{B(x)}$  by checking that their values agree on 121 different points in  $E(K)$ . The ratio of these two functions is a



map from  $E \rightarrow \mathbb{P}^1$  of degree at most 120, and since this function takes on the value 1 at 121 different points, we have that  $\psi(x, y) = \frac{A(x)}{B(x)}$ .

At this point, we use Nils Bruin's Magma implementation of elliptic curve Chabauty to find all points  $R \in E(K)$  with  $\psi(R) \in \mathbb{P}^1(\mathbb{Q})$ . There are five such points in  $E(K)$ , and the preimages of these points give eight points in  $C_K(K)$ . The only one of these points that is rational is  $(0 : 1 : 0 : 0)$ , and this corresponds to  $x^5 - 5x + 12$ .

Now, we consider the case where  $K = \mathbb{Q}[\sqrt[5]{18}]$ . The polynomial  $x^5 + 750x + 3750$  is equivalent to  $x^5 + (6/5)x + (6/5)$  and so we take  $t = 6/5$  in the calculations in the previous section. The field  $L$  is  $K[\sqrt[5]{18}, \sqrt{5}]$ . The elliptic curve  $E$  is the quadratic twist of  $E_0$  by  $\frac{-5+\sqrt{5}}{2}$ . Computing the 2-Selmer group of  $E$  over  $L$  requires knowing the class group of a degree 3 extension of  $L$ , and we are able to determine this only by assuming GRH. A 140-hour computation in Magma determines that the 2-Selmer rank of  $E/L$  is 2. Most of this time is a systematic search for 2-adic points on the curve. There are 5 rational points on  $C_K$ , and their images on  $E$  generate a subgroup of rank 1. Without a second point of infinite order, we cannot apply elliptic curve Chabauty. Directly point searching on  $E$  is not likely to be effective, and explicitly computing the 2-covers of  $E$  is not feasible computationally.

By taking the curve  $C_K$  and computing its automorphism group modulo several primes, we are led to suspect that  $C_K$  has 8 automorphisms defined over  $L$ . This can be confirmed using the method in [16], Subsection 7.5. Applying these automorphisms to the points on  $C_K$  give 16 points in  $C_K(L)$  (in two orbits). The images of these points on  $E$  generate a subgroup of rank 2. Notably, the points on  $C_K$  corresponding to trinomials with  $t = 0$  are torsion points on  $E$ , and  $E(L) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ . To compute the map  $\psi : E \rightarrow \mathbb{P}^1$  we suppose that it has the form  $\frac{A(x)+yB(x)}{C(x)}$ , where  $\deg A, C \leq 60$  and  $\deg B \leq 58$ . This time, we parametrize the quadric surface containing  $C_K$  and using this parametrization create points on  $C_K$  over cubic extensions of  $L$ . To find the coefficients of  $A(x)$ ,  $B(x)$  and  $C(x)$ , we exactly solve a  $185 \times 181$  linear system (over  $L$ ). Solving this system in Magma takes 145 hours and the list of the resulting coefficients requires more than 1 megabyte. As in the previous case, we verify that  $\frac{A(x)+yB(x)}{C(x)}$  is equal to  $\psi$  by verifying that  $\frac{A(x)+yB(x)}{C(x)}$  has degree 60 and that these two maps give the same values on 121 different points on  $E$ .

At this point we can apply elliptic curve Chabauty, which requires only 127 seconds to run. We find that there are nine points in  $E(L)$  that have rational image under  $\psi$ , and there are 16 points in  $C_K(L)$  that map to these. Of these, only 5 are rational. These are  $(0 : 1 : 0 : 0)$ , corresponding to  $x^5 + (6/5)x + (6/5)$ ,  $(-168/55 : 9/11 : 19/11 : 1)$ , corresponding to  $x^5 - 18$ ,  $(36/35 : -30/7 : 24/7 : 1)$ , corresponding to  $x^5 - 432$ ,  $(-22/15 : -7/6 : -5/4 : 1)$ , corresponding to  $x^5 - 324$ , and  $(-8/65 : 20/39 : -16/39 : 1)$ , corresponding to  $x^5 - 24$ .  $\square$

## 6. NUMBER FIELDS WITH MORE THAN ONE TRINOMIAL DEFINING THEM

Taking the equation of the curve from Proposition 4 and eliminating the parameter  $t$  yields the surface

$$\begin{aligned} X : & 20a^3cd^2 + 15a^3d^3 + 128a^2b^2d^2 + 128a^2bc^2d + 240a^2bcd^2 - 100a^2bd^3 + 32a^2c^4 \\ & + 320a^2c^3d + 700a^2c^2d^2 + 250a^2cd^3 - 128ab^3cd - 480ab^3d^2 - 64ab^2c^3 \\ & - 720ab^2c^2d - 600ab^2cd^2 - 500ab^2d^3 - 160abc^4 - 600abc^3d - 1500abc^2d^2 \\ & - 2500abcd^3 + 400ac^5 + 2000ac^4d + 2500ac^3d^2 + 1280b^4cd + 1600b^4d^2 + 640b^3c^3 \\ & + 4000b^3c^2d + 2000b^3cd^2 + 800b^2c^4 + 2000b^2c^3d = 0. \end{aligned}$$

If  $(a : b : c : d)$  is a point on  $X$ , and  $\alpha$  is a root of  $x^5 + tx + t$ , where  $t = \frac{5a^2 - 50ab}{32bd + 16c^2 + 40cd}$ , then

$$a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4$$

is also a root of a trinomial in  $\mathbb{Q}[\alpha]$ , where  $e = \frac{5a}{4t}$ .

The surface  $X$  is (very) singular. It has four lines on it:

- $a = 10b, c = -(3/5)b$ . These points correspond to  $t = 0$ .
- $a = b = 0$ . These also correspond to  $t = 0$ .
- $b = (21/32)d, c = (-3/4)d$ . These correspond to  $t = \infty$ .
- $a = (125/16)d, c = (-5/4)d$ . These correspond to  $t = -3125/256$  (for which  $x^5 + tx + t$  is reducible).
- $c = d = 0$ . These points are singular.

The family where  $d = tc$  is a fibration of  $X$  by curves that (for all but finitely many  $t$ ) have genus 2. Viewing  $X$  as a hyperelliptic curve over  $\mathbb{Q}(t)$ , we find that it has a model in the form  $y^2 = f(x, t)$ , and  $y = 0$  is a non-hyperelliptic genus 4 curve. This implies that a non-singular model  $\tilde{X}$  of  $X$  is a K3 surface. This implies in addition that  $X$  has an automorphism of order 2 defined over  $\mathbb{Q}$ .

In addition, the surface has at least seven rational curves:

- $a = -\frac{3}{100}s^4 - \frac{1}{5}s^3 + s^2, b = \frac{3}{100}s^3 + \frac{1}{5}s^2 - s, c = 0, d = \frac{32}{125}s^2 + \frac{24}{25}s + \frac{16}{5}$ .
- $a = \frac{7}{2000}s^4 + \frac{1}{100}s^3 + \frac{1}{4}s^2 + s, b = -\frac{7}{2000}s^3 - \frac{1}{100}s^2 - \frac{1}{4}s - 1, d = \frac{8}{625}s^2 + \frac{2}{125}s - \frac{4}{25}, c = -\frac{5}{2}d$ .
- $a = -\frac{1}{250}s^3 + \frac{2}{25}s^2 - \frac{1}{2}s + 1, b = -\frac{1}{250}s^2 + \frac{1}{10}, d = -\frac{32}{625}s + \frac{16}{125}, c = -\frac{5}{4}d$ . This curve is the one that appears in Theorem 3.
- $a = 0, b = -\frac{1}{2}s^2 - \frac{5}{4}s, c = s, d = 1$ .
- $a = -5s^2 - \frac{25}{2}s, b = -\frac{1}{2}s^2 - \frac{5}{4}s, c = s, d = 1$ . This curve is contained in the singular locus of  $X$ .
- The image on  $X$  of the family of trinomials defining  $\mathbb{Q}[(r^3(r+1)(r-1)^4)^{1/5}]$  given in [20]. For this curve,  $c$  and  $d$  have degree 25, while  $a$  and  $b$  have degree 24.
- The image of the previous curve under the automorphism. For this curve,  $a, b, c$  and  $d$  all have degree 49.

We conclude by raising a number of questions:

- (1) What are equations for  $\tilde{X}$ ?
- (2) What is the rank of the Néron-Severi group of  $\tilde{X}$ ?
- (3) Above we give five rational curves on  $X$ . It is conjectured that there are infinitely many rational curves (defined over  $\mathbb{C}$ ) on any K3 surface. Is this true for  $\tilde{X}$ ? Are the  $\mathbb{Q}$ -rational points on  $\tilde{X}$  Zariski dense?

- (4) Are there infinitely many number fields  $K$  that contain roots of three inequivalent trinomials?
- (5) Are there infinitely many number fields  $K/\mathbb{Q}$  whose Galois closure has Galois group  $D_{10}$  that contain the roots of two trinomials? (We know of 3 examples.)

## REFERENCES

- [1] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).
- [2] Andrew Bremner and Blair K. Spearman. Cyclic sextic trinomials  $x^6 + Ax + B$ . *Int. J. Number Theory*, 6(1):161–167, 2010.
- [3] Thomas Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.
- [4] Nils Bruin. Chabauty methods using elliptic curves. *J. Reine Angew. Math.*, 562:27–49, 2003.
- [5] Nils Bruin and Noam D. Elkies. Trinomials  $ax^7 + bx + c$  and  $ax^8 + bx + c$  with Galois groups of order 168 and  $8 \cdot 168$ . In *Algorithmic number theory (Sydney, 2002)*, volume 2369 of *Lecture Notes in Comput. Sci.*, pages 172–188. Springer, Berlin, 2002.
- [6] Lucia Caporaso, Joe Harris, and Barry Mazur. Uniformity of rational points. *J. Amer. Math. Soc.*, 10(1):1–35, 1997.
- [7] Vladimir Dokchitser. Root numbers of non-abelian twists of elliptic curves. *Proc. London Math. Soc. (3)*, 91(2):300–324, 2005. With an appendix by Tom Fisher.
- [8] W. L. Edge. Bring’s curve. *J. London Math. Soc. (2)*, 18(3):539–545, 1978.
- [9] D. W. Erbach, J. Fisher, and J. McKay. Polynomials with  $\mathrm{PSL}(2, 7)$  as Galois group. *J. Number Theory*, 11(1):69–75, 1979.
- [10] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. Math.*, 73(3):349–366, 1983.
- [11] Walter Feit. Some consequences of the classification of finite simple groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 175–181. Amer. Math. Soc., Providence, R.I., 1980.
- [12] Felix Klein. *Lectures on the icosahedron and the solution of equations of the fifth degree*. Dover Publications, Inc., New York, N.Y., revised edition, 1956. Translated into English by George Gavin Morrice.
- [13] Gunter Malle. Polynomials for primitive nonsolvable permutation groups of degree  $d \leq 15$ . *J. Symbolic Comput.*, 4(1):83–92, 1987.
- [14] B. Heinrich Matzat. *Konstruktive Galoistheorie*, volume 1284 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.
- [15] Rick Miranda. *Algebraic curves and Riemann surfaces*, volume 5 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1995.
- [16] Jeremy Rouse and David Zureick-Brown. Elliptic curves over  $\mathbb{Q}$  and 2-adic images of Galois. *Research in Number Theory*, 1(12):1–34, 2015.
- [17] Jean-Pierre Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.
- [18] Jean-Pierre Serre. *Topics in Galois theory*, volume 1 of *Research Notes in Mathematics*. Jones and Bartlett Publishers, Boston, MA, 1992. Lecture notes prepared by Henri Damon [Henri Darmon], With a foreword by Darmon and the author.
- [19] Blair K. Spearman and Kenneth S. Williams. On solvable quintics  $X^5 + aX + b$  and  $X^5 + aX^2 + b$ . *Rocky Mountain J. Math.*, 26(2):753–772, 1996.
- [20] Blair K. Spearman and Kenneth S. Williams. Pure quintic fields defined by trinomials. *Rocky Mountain J. Math.*, 30(1):371–391, 2000.
- [21] Andrew V. Sutherland. Computing images of galois representations attached to elliptic curves. Preprint.
- [22] H. Weber. *Lehrbuch der Algebra I*. Braunschweig F. Vieweg, 1898.

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